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HIGH ACCURACY SIMULATIONS OF RESISTIVITY LOGGING INSTRUMENTS USING A SELF-ADAPTIVE GOAL-ORIENTED *HP*-FEM

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Summary. *We describe the development and application of a 2D Finite Element (FE) self-adaptive hp goal-oriented algorithm for elliptic and electromagnetic problems. The algorithm delivers (without any user interaction) a sequence of optimal hp-grids that converges exponentially in terms of a user-prescribed* quantity of interest *with respect to the CPU time. We illustrate the efficiency of the method with 2D numerical simulations of resistivity logging instruments.*

1 INTRODUCTION

During the last decades, different algorithms intended to generate *optimal* grids for solving relevant engineering problems have been designed and implemented. Among those algorithms, a self-adaptive, energy-norm based, *hp*-Finite Element (FE) refinement strategy has been developed at the Institute for Computational Engineering and Sciences (ICES) of The University of Texas at Austin. The strategy produces automatically a sequence of *hp*-meshes that delivers *exponential convergence rates* in terms of the *energy-norm* error against the number of unknowns (as well as the CPU time), independently of the number and type of singularities of the problem. Thus, it provides high accuracy approximations of solutions corresponding to a variety of engineering applications. Furthermore, the self-adaptive strategy is *problem independent*, and it can be applied to FE discretizations of H^1-, $H(\mathbf{curl})$-, and $H(div)$-spaces, as well as to nonlinear problems (see [2, 7] for details).

However, the energy-norm is a quantity of limited relevance for most engineering applications, especially when a particular objective is pursued, for instance, to simulate the electromagnetic response of geophysical resistivity logging instruments in a borehole environment. In these instruments, the amplitude of the measurement (for example, the electric field) is typically several orders of magnitude smaller at the receiver antennas than at the transmitter antennas. Thus, small relative errors of the solution in the energy-norm *do not* imply small relative errors of the solution at the receiver antennas. Indeed, it is not uncommon to construct energy-norm based adaptive grids delivering a relative error in the energy-norm below 1%, while the solution at the receiver antennas still exhibits a relative error above 1000% (see [4]).

Consequently, a self-adaptive strategy is needed to approximate a specific feature of the solution. Refinement strategies of this type are called *goal-oriented* adaptive algorithms [3], and are based on minimizing the error of a prescribed *quantity of interest* mathematically expressed in terms of a linear functional.

2 SELF-ADAPTIVE GOAL-ORIENTED *hp*-FEM

We are interested in solving the following variational problem:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{E}_D + \mathbf{V} \\ b(\mathbf{E},\mathbf{F}) = f(\mathbf{F}) \quad \forall \mathbf{F} \in \mathbf{V} \ , \end{array} \right. \tag{2.1}$$

where $\mathbf{E}_D$ is a lift of the essential (Dirichlet) BC, $\mathbf{V}$ is a Hilbert space, $\mathbf{V}'$ is an anti-linear and continuous functional on $\mathbf{V}$, and b is a sesquilinear form. More precisely, we have:

$$b(\mathbf{E},\mathbf{F}) = D_1 a(\mathbf{E},\mathbf{F}) - D_2 c(\mathbf{E},\mathbf{F}) \ , \tag{2.2}$$

where sesquilinear forms a and c are assumed to be Hermitian, continuous and $\mathbf{V}$-coercive, and $D_1, D_2 \geq 0$. We define an "energy" inner product on $\mathbf{V}$ as $(\mathbf{E},\mathbf{F}) := D_1 a(\mathbf{E},\mathbf{F}) + D_2 c(\mathbf{E},\mathbf{F})$, with the corresponding (energy) norm denoted by $\|\mathbf{E}\|$.

Given an *hp*-FE subspace $\mathbf{V}_{hp} \subset \mathbf{V}$, we discretize (2.1) as follows:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{E}_{hp} \in \mathbf{E}_D + \mathbf{V}_{hp} \\ b(\mathbf{E}_{hp},\mathbf{F}_{hp}) = f(\mathbf{F}_{hp}) \quad \forall \mathbf{F}_{hp} \in \mathbf{V}_{hp} \ . \end{array} \right. \tag{2.3}$$

The objective of goal-oriented adaptivity is to construct an optimal *hp*-grid, in the sense that it minimizes the problem size needed to achieve a given tolerance error for a given *quantity of interest* L, with L denoting a linear and continuous functional. By recalling the linearity of L, we have $L(\mathbf{E}) - L(\mathbf{E}_{hp}) = L(\mathbf{E} - \mathbf{E}_{hp}) = L(\mathbf{e})$, where $\mathbf{e} = \mathbf{E} - \mathbf{E}_{hp}$ denotes the error function. By defining the residual $\mathbf{r}_{hp} \in \mathbf{V}'$ as $\mathbf{r}_{hp}(\mathbf{F}) = f(\mathbf{F}) - b(\mathbf{E}_{hp},\mathbf{F}) = b(\mathbf{E} - \mathbf{E}_{hp},\mathbf{F}) = b(\mathbf{e},\mathbf{F})$, we look for the solution of the *dual problem*:

$$\left\{ \begin{array}{l} \text{Find } \bar{\mathbf{W}} \in \mathbf{V} \\ b(\mathbf{F},\mathbf{W}) = L(\mathbf{F}) \quad \forall \mathbf{F} \in \mathbf{V} \ . \end{array} \right. \tag{2.4}$$

Using the (generalized) Lax-Milgram theorem we conclude that problem (2.4) has a unique solution in $\mathbf{V}$. The solution $\mathbf{W}$ is usually referred to as the *influence function*.

By discretizing (2.4) via, for example, $\mathbf{V}_{hp} \subset \mathbf{V}$, we obtain:

$$\left\{ \begin{array}{l} \text{Find } \bar{\mathbf{W}}_{hp} \in \mathbf{V}_{hp} \\ b(\mathbf{F}_{hp},\mathbf{W}_{hp}) = L(\mathbf{F}_{hp}) \quad \forall \mathbf{F}_{hp} \in \mathbf{V}_{hp} \ . \end{array} \right. \tag{2.5}$$

Definition of the dual problem plus the Galerkin orthogonality for the original problem imply the final representation formula for the error in the quantity of interest, namely,

$$L(\mathbf{e}) = b(\mathbf{e}, \mathbf{W}) = b(\mathbf{e}, \mathbf{W} - \mathbf{F}_{hp}) = \tilde{b}(\mathbf{e}, \epsilon)\,, \tag{2.6}$$

where $\epsilon = \mathbf{W} - \mathbf{F}_{hp}$. At this point, $\mathbf{F}_{hp} \in \mathbf{V}_{hp}$ is arbitrary, and $\tilde{b}(\mathbf{e}, \epsilon) = b(\mathbf{e}, \bar{\epsilon})$ denotes the bilinear form corresponding to the original sesquilinear form.

Notice that, in practice, the dual problem is solved not for $\mathbf{W}$ but for its complex conjugate $\bar{\mathbf{W}}$ utilizing the bilinear form and *not* the sesquilinear form. The linear system of equations is factorized only once, and the extra cost of solving (2.5) reduces to only one backward and one forward substitution (if a direct solver is used).

Once the error in the quantity of interest has been determined in terms of bilinear form $\tilde{b}$, we wish to obtain a sharp upper bound for $|L(\mathbf{e})|$ that depends upon the mesh parameters (element size h and order of approximation p) *only locally.* Then, a self-adaptive algorithm intended to minimize this bound will be defined.

First, using a procedure similar to the one described in [2], we approximate $\mathbf{E}$ and $\mathbf{W}$ with *fine grid* functions $\mathbf{E}_{\frac{h}{2},\,p+1}$, $\mathbf{W}_{\frac{h}{2},\,p+1}$, which have been obtained by solving the corresponding linear system of equations associated with the FE subspace $\mathbf{V}_{\frac{h}{2},\,p+1}$. Next, we bound the error in the quantity of interest by a sum of element contributions. Let b_K denote a contribution from element K to sesquilinear form b. It then follows that

$$|L(\mathbf{e})| \approx |b(\mathbf{E}_{\frac{h}{2},p+1} - \mathbf{E}_{h,p}, \mathbf{W}_{\frac{h}{2},p+1} - \mathbf{W}_{h,p})| \le \sum_K |b_K(\mathbf{E}_{\frac{h}{2},p+1} - \mathbf{E}_{h,p}, \mathbf{W}_{\frac{h}{2},p+1} - \mathbf{W}_{h,p})|\,, \tag{2.7}$$

where summation over K indicates summation over elements. Using the projection based interpolation operator $\mathbf{\Pi}_{hp}$ defined in [1], and applying Cauchy-Schwartz inequality, we obtain:

$$|L(\mathbf{e})| \preceq \sum_K \|\tilde{\mathbf{e}}\|_K \|\tilde{\boldsymbol{\epsilon}}\|_K\,, \tag{2.8}$$

where $\tilde{\mathbf{e}} = \mathbf{E}_{\frac{h}{2},p+1} - \mathbf{\Pi}_{hp}\mathbf{E}_{\frac{h}{2},p+1}$, $\tilde{\boldsymbol{\epsilon}} = \mathbf{W}_{\frac{h}{2},p+1} - \mathbf{\Pi}_{hp}\mathbf{W}_{\frac{h}{2},p+1}$, and $\|\cdot\|_K$ denotes energy-norm $\|\cdot\|$ restricted to element K. Finally, we define a self-adaptive algorithm intended to minimize estimate (2.8) (see [5] for details).

3 NUMERICAL RESULTS

In Fig. 1, we compare the self-adaptive goal-oriented and energy-norm based *hp*-algorithms, for the simulation of a Logging-While-Drilling instrument described in [6]. In order to resolve the problem it is essential the use of goal-oriented adaptivity.

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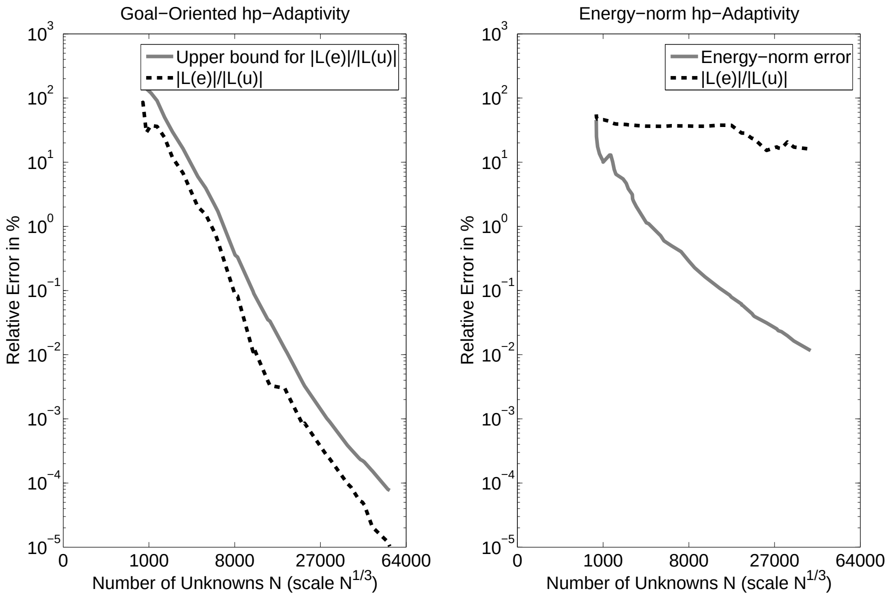

Figure 1: Logging-While-Drilling problem. Left panel: convergence behavior obtained with the self-adaptive goal-oriented *hp*-FEM shows exponential convergence rates for estimate (2.8) (solid curve) used for optimization. The dashed curve describes the relative error in the quantity of interest. Right panel: convergence behavior obtained with the self-adaptive energy-norm *hp*-FEM shows exponential convergence rates for the energy-norm. The dashed curve describes the relative error in the quantity of interest.

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