Feasibility study for 2D frequency-dependent electromagnetic sensing through casing

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ABSTRACT

We perform a numerical-sensitivity study of various physical phenomena of interest for borehole resistivity logging applications in steel-cased wells. Specifically, we analyze the sensitivity of through-casing measurements for detecting water invasion, shale-laminated sands, and electrical anisotropy in the formation. In addition, we study the influence of the frequency of operation on through-casing resistivity measurements. The sensitivity analysis is performed using a highly accurate and reliable numerical method based on a 2D self-adaptive, goal-oriented, high-order, finite-element method (FEM). This method can be applied to simulate all types of resistivity logging measurements, including normal/laterolog, induction, and through-casing resistivity measurements. Results quantify the effects of several physical phenomena that can be sensed through casing and that can be measured with accurate sensors. We find that water invasion and shale-laminated sands behind casing can be detected and accurately quantified at frequencies below 10 Hz. On the other hand, measurements are almost insensitive to electrical anisotropy behind casing, but otherwise highly sensitive to frequency variations. Our simulations indicate that a frequency in the range of 3–30 Hz is the most adequate for maximizing the sensitivity of through-casing measurements with respect to spatial variations of electrical conductivity within the formation.

INTRODUCTION

Steel casing is the standard means to complete wells drilled into hydrocarbon reservoirs. Although steel casing provides mechanical competence that prevents the well from collapsing, especially across soft-rock formations, the presence of electrically conductive casing poses significant challenges to probing electrical properties behind the casing. The high electrical conductivity of casing causes the amplitude of electromagnetic (EM) fields to decay exceedingly fast, thereby inhibiting the accurate sensing of formation electrical conductivity behind casing. Not until the last decade were reliable and accurate instrumentation systems introduced to probe electrical resistivity behind casing. Interpretation of resistivity logging measurements behind casing was finally possible mainly because of the following advances:

1) Fundamental theoretical work to quantify EM fields measured in the presence of steel casing (Kaufman, 1990) and the design of reliable calibrated instruments (Vail et al., 1993)
2) Advances in the manufacturing of EM sensors that minimize the noise-to-signal ratio and enable reliable measurement of voltages in the nanovolt range (An example of such an advance is the cased-hole formation resistivity (CHFR) tool, introduced by Schlumberger.)
3) Advances in computational EM necessary to simulate numerically the resistivity logging measurements acquired in the high-conductivity environment of cased wells (In this area, the contributions by Schenkel and Morrison [1994] are remarkable for quantifying the possibility of crosswell resistivity surveys if one well is cased, as are contributions by Wu and Habashy [1994], who studies the response of a through-casing resistivity instrument as a function of frequency.)

More recent contributions to the field of through-casing measurements include the works of Singer and Strack (1994) who developed accurate modeling solutions, and Pardo et al. (2006b) in the area of computational modeling.

The central objective of this paper is to quantify the sensitivity of through-casing resistivity measurements to several physical effects often encountered while probing rock formations behind casing. To that end, we use a highly accurate numerical simulation software designed specifically for that purpose.
Our numerical methodology is based on a self-adaptive, goal-oriented hp finite-element method (FEM) that converges exponentially in terms of a user-prescribed quantity of interest versus the problem size (as well as CPU time). The method combines elements of different size h and various polynomial orders of approximation p and incorporates a self-adaptive strategy to automatically generate an optimal distribution of h and p throughout the grid intended to approximate the quantity of interest. We have successfully applied this highly accurate and very reliable adaptive algorithm to a variety of resistivity logging devices, including simple through-casing instruments, induction logging, and normal/laterolog instruments. A detailed description of the numerical methodology can be found in Pardo et al. (2006a) and Pardo et al. (2006b). For the problems considered in this paper, our numerical methodology guarantees relative errors below 1%.

This paper is a continuation of the results presented in Pardo et al. (2006b). Here, we consider more challenging problems, and we analyze physically relevant effects through casing, such as water invasion, detection of electrical anisotropy, detection of thin layers, and influence of frequency on the measurements.

First, we introduce the logging applications of interest that are driving our objectives. These resistivity logging applications are governed by Maxwell’s equations, which we describe in the next section, along with the corresponding variational formulation for axisymmetric problems. Then we illustrate our findings with numerical results. Conclusions are presented thereafter. Finally, Appendix A contains validation results for our finite-element software. Here, we compare results obtained with our self-adaptive, goal-oriented hp-refinement strategy against those obtained with a semianalytic 1D radial software for a number of model problems.

All problems and applications considered in this article are assumed to be axisymmetric; that is, the geometry, material properties, and distribution of antennas are axial symmetric with respect to the axis of the borehole.

THROUGH-CASING APPLICATIONS

In this section, we describe three through-casing axisymmetric model problems in a borehole environment that are of great interest in a variety of logging applications. In all problems, we consider a uniform 1.27-cm-thick casing with resistivity equal to 2.3 × 10⁻⁷ Ωm and 85 relative permeability. Casing is surrounded by a 5-cm layer of cement with a resistivity of 2 Ωm. The radius of the borehole is 10 cm, and it is filled with a material of resistivity equal to 1 Ωm.

Measurements are based on the use of one transmitter and two receiver antennas, which are located 1.25 and 1.5 m above the transmitter, respectively. We simulate the first difference of the vertical component of the electric field at receiving coils l₁ and l₂ of radius a = 0.1 cm divided by the (vertical) distance Δz between them and the perimeter of the coils, i.e.,

$$\frac{\int_{l_1} E_z(l_1) dl - \int_{l_2} E_z(l_2) dl}{[(\Delta z)(2\pi a)]} = \frac{1}{\Delta z} [E_z(l_1) - E_z(l_2)].$$

We consider three different formations, as described in Figures 1–3, respectively. In the first application (Figure 1), the objective is to study the sensitivity of the quantity of interest given by equation 1.
with respect to variations of frequency (from 3 to 90 Hz) and water invasion (of resistivity equal to 0.8 Ωm) in the target (oil-bearing) zone. In our second model (Figure 2), we study the possibility of sensing through casing the different layers of laminated shales and sands existing in the formation. We also study the laminated sand sensitivity to variations of frequency from 0.3 to 3 Hz and the effect of water invasion (resistivity = 0.8 Ωm) into the sand layers. In our third problem (Figure 3), we study the anisotropy and water invasion effects with respect to variations of frequency (3–30 Hz).

We assume a casing of uniform thickness and resistivity. Although this is almost never the case in practical applications (because of joints and corrosion effects in the casing), it is possible to use calibrated tools to account for such effects. A design of a calibrated tool is shown, for instance, in Pardo et al. (2006b); for simplicity of presentation, we do not consider calibrated tools in this paper.

After the well has been cemented and the casing has been set, there still may exist fluid movement within the formation, either because of gravity segregation and capillary equilibrium forces acting for months in response to previous water injection or because of oil extraction or water injection through perforations that hydraulically connect the borehole with the formation.

**MAXWELL’S EQUATIONS**

All these problems are governed by Maxwell’s equations. Assuming a time-harmonic dependence of the form $e^{j\omega t}$, where $j = \sqrt{-1}$ is the imaginary unit, $t$ denotes time, and $\omega$ is angular frequency, Maxwell’s equations can be written as

$$\begin{align*}
\nabla \times \mathbf{H} &= (\hat{\mathbf{\sigma}} + j\omega\hat{\mathbf{\mu}})\mathbf{E} + \mathbf{J}^{\text{imp}} \\
\nabla \times \mathbf{E} &= -j\omega\mathbf{\mu}\mathbf{H} - \mathbf{M}^{\text{imp}} \\
\nabla \cdot (\hat{\mathbf{\varepsilon}}\mathbf{E}) &= \rho \\
\nabla \cdot (\hat{\mathbf{\mu}}\mathbf{H}) &= 0.
\end{align*}$$

(2)

Here, $\mathbf{H}$ and $\mathbf{E}$ denote the magnetic and electric fields, respectively; $\mathbf{J}^{\text{imp}}$ is a prescribed, impressed electric current density; $\mathbf{M}^{\text{imp}}$ is a prescribed, impressed magnetic current density; tensors $\hat{\mathbf{\varepsilon}}$, $\hat{\mathbf{\mu}}$, and $\hat{\mathbf{\sigma}}$ stand for dielectric permittivity, magnetic permeability, and electrical conductivity of the medium, respectively; and $\rho$ denotes the electric charge distribution. We assume that the determinants of $\hat{\mathbf{\mu}}$ and $\hat{\mathbf{\sigma}} + j\omega\hat{\mathbf{\varepsilon}}$ are nonzero.

**Source antennas**

Toroid antennas are modeled by prescribing an impressed volume magnetic current $\mathbf{M}^{\text{imp}}$. Using cylindrical coordinates ($\rho, \phi, z$), and to avoid the dependence upon the geometrical dimensions of the toroid, we select $\mathbf{M}^{\text{imp}} = \delta(z)\delta(\rho - a)I_0\phi_0\hat{\mathbf{z}}$, where $I_0 = 1/(\pi a^2) \approx 0.003$ and $a = 0.1$ m is the radius of the coil. The imposed magnetic current on the toroidal coil is equivalent to that induced by an electric excitation using a vertical electrical dipole, also known as Hertzian dipole, with current equal to $(\sigma + j\omega e)$ amperes. The corresponding magnetic far-field solution in homogeneous media is given by (Lovell, 1993)

$$\mathbf{H} = \hat{\phi}(\sigma + j\omega e)jke^{-jkd}$$

$$\frac{1}{4\pi d} \left[ 1 - \frac{j}{kd} \right] \frac{\rho}{d},$$

(3)

where $k = \sqrt{\mu\varepsilon - j\omega\sigma}$ is the square root of the wavenumber and $d$ is the distance between the center of the transmitter and the receiver.

In practical applications, the effective magnetic moment of a toroid depends not only on the circulating electrical current but also on the frequency of operation, number of wire turns, cross-sectional area, perimeter, and permeability of the core material. The simulation results described in this paper have not been normalized to reflect measurement conditions. Such a step can only be performed if provided with the pertinent geometric and physical properties of the measurement system.

**Variational formulation**

For axisymmetric fields, the corresponding variational formulation in terms of the azimuthal component of the magnetic field $H_\phi = (0, H_\phi, 0)$ is given by

$$\begin{align*}
\text{Find } H_\phi &\in H_\phi,D + \tilde{H}_D(Y) \text{ such that:} \\
\int_Y [(\hat{\mathbf{\sigma}}_{p,c} + j\omega\hat{\mathbf{\varepsilon}}_{p,c})^{-1} \nabla \times \mathbf{H}] \cdot (\nabla \times \tilde{\mathbf{F}}_\phi)dV \\
&+ j\omega\int_Y (\tilde{\mathbf{\mu}}_\phi\mathbf{H}_\phi) \cdot \tilde{\mathbf{F}}_\phi dV \\
&= -\int_Y M^{\text{imp}} \tilde{\mathbf{F}}_\phi dV + \int_S M^{\text{imp}} \tilde{\mathbf{F}}_\phi dS \\
\forall \mathbf{F}_\phi &\in \tilde{H}_D(Y),
\end{align*}$$

(4)

Figure 3. Geometry of problem III. Measurements based on one transmitter (toroidal) antenna and two receiver antennas. Formation composed of a borehole, casing, cement, background layer (shale), and target zone formed of oil-, oil-water-, and water-saturated layers, all separated by shale layers. Resistivity of casing: $2.3 \times 10^{-7}$ Ωm.
where \( Y \) is our domain of interest; \( I_{aD} \) represents the Dirichlet data, typically \( H_{aD} = 0; F_{aD} = (0, F_{aD}) \in \mathcal{H}(Y) \) is a test function, 
\[
\tilde{H}_a(Y) = \{ H_a : (0, H_{aD}) \in \mathcal{H}(Y) \}
\]
\( \text{curl} Y = \{ [H_a \in L^2(Y) : \text{curl } Y = \{ H_a \} \text{ is a test function, } \} \}
\] 
\( \text{and permittivity tensors, respectively; and } \mu_a, \epsilon_a \) are the meridional components of the conductivity and permittivity tensors, respectively; and \( \mu_a, \epsilon_a \) is the azimuthal component of the magnetic permeability tensor. For a more detailed derivation of the variational formulation, see Pardo et al. (2006a).

Boundary conditions

We consider a bounded computational domain \( Y \). A variety of boundary conditions (BCs) can be imposed on the boundary \( \partial Y \), such that the difference between the solution of such a problem and the solution of the original problem defined over the whole space is small. Because the EM fields and their derivatives decay exponentially in the presence of lossy media (nonzero conductivity), we impose a homogeneous Dirichlet BC on the boundary of a sufficiently large computational domain. We select for all of the simulations a domain 3000 m in the horizontal direction and 6000 m in the vertical direction (Figure 1).

NUMERICAL RESULTS

Below, we obtain numerical results for the three problems of interest introduced above using the self-adaptive, goal-oriented \( hp \)-FEM described in Pardo et al. (2006a). In particular, we analyze a number of computer-generated logs displaying the amplitude and phase of the normalized first vertical difference of \( E_z \) (equation 1) as a function of the vertical position (in meters) of the center of the two receiver antennas.

Problem I: Water-invaded hydrocarbon reservoir

In Figure 4, we display various logs corresponding to problem I at 3 Hz and 90 Hz. Solid curves indicate simulated measurements without water invasion. Dashed curves correspond to simulated measurements with a 50-cm, pistonlike water invasion within the target zone. Results are very sensitive to the target (oil-bearing) layer, which can be identified clearly from the logging measurements. For the 3-Hz case, we observe relative differences in the simulated measurements as large as 100%.

As we increase the frequency, the sensitivity with respect to the target layer diminishes. We also observe that the water invasion through casing increases the measured signal by as much as one order of magnitude for the 3-Hz case. Sensitivity with respect to water invasion significantly decreases as we increase the frequency. Shoulder effects are more noticeable at large frequencies. From these results, we conclude that frequencies in the range of 3 Hz are adequate to monitor through-casing water invasion.

Figure 5 displays the frequency dependence of the simulated measurements for problem I. At frequencies below 5 Hz, we observe an almost constant response. A rapid amplitude decay is observed at larger frequencies, from 10 to 100 Hz. This decay occurs because EM fields decay exponentially with respect to frequency once the exponential term becomes the dominant part of the solution (in this case, it occurs at 10 Hz).

Problem II: Laminated sands

In Figure 6, we display the amplitude and phase of the quantity of interest (see equation 1) as a function of the vertical position (in meters) of the receiver antennas. Different curves correspond to various radii of piston-like water invasion into the sand layers: no invasion, 20-cm invasion, 50-cm invasion, and 80-cm invasion. Through-casing measurements are sensitive to the layers of laminated shales and sands existing in the formation. More precisely, we can clearly identify the vertical position of the 10 layers of sands and shales because the corresponding amplitude of the measurements varies by up to a factor of two. This sensitivity is present in terms of both the amplitude and the phase of the simulated measurements.
As we invade the sands with water, we observe a considerable increase of the measured signal. We also observe that, as we invade the sands with water, the identification of sand and shale layers existing on the formation becomes unclear because in our model, water-invaded sands have a resistivity similar to that of shale layers. For the cases of 50-cm and 80-cm water invasion, we notice a higher response on the upper part of the sand-shale layers. This is because of the absence of vertical symmetry in the considered logging instrument. We notice a larger amplitude response when both transmitter and receivers are located within the target zone (conductive layers).

In Figure 7, we display the same results as the ones displayed in Figure 6 but at a lower frequency, 0.3 Hz. We obtain identical results in terms of the amplitude of the simulated measurements, consistent with those of a low-frequency regime. Phase differences, however, are roughly one order of magnitude smaller, as physically expected because of the use of a lower frequency. From these results, we conclude that frequencies below 3 Hz do not provide the additional sensitivity needed to estimate water invasion and distinguish between sand and shale layers. On the contrary, phase sensitivity is seriously deteriorated because of the low frequency.

**Problem III: Anisotropy and invasion**

In Figure 8, we display the amplitude and phase of the quantity of interest (equation 1) as a function of the vertical position (in meters) of the receiver antennas. Different curves correspond to the presence (or absence) of water invasion and anisotropy in the formation. Simulated measurements through casing are sensitive to the different materials within the formation. Water invasion effects are significant on the phase of the simulated measurements corresponding to the oil- and oil-water-saturated layers. In terms of the amplitude of the measurements, water-invasion effects are less noticeable (below 5%).

Anisotropy effects are subtle and difficult to identify from the measurements, mainly because leakage in a uniform casing occurs almost exclusively in the radial direction. However, we notice a slight difference on the shoulder effects corresponding to anisotropic formations, probably because of nonradial current leakage in the proximity of material interfaces.

Figure 7. Simulated through-casing measurements for problem II. (a) Amplitude and (b) phase of the first vertical difference of $E_z$ (normalized) against the vertical position (in meters) of the receiver antennas. Different curves correspond to various radii of water invasion into the sand layers. Operating frequency: 0.3 Hz.

Figure 8. Simulated through-casing measurements for problem III. (a) Amplitude and (b) phase of the first vertical difference of $E_z$ (normalized) against the vertical position (in meters) of the receiver antennas. Different curves correspond to the presence (or not) of water invasion and possible anisotropy within the shale layers. No invasion, no anisotropy; 15-cm water invasion, no anisotropy; no water invasion, shale anisotropy; 15-cm water invasion, shale anisotropy. Operating frequency: 3 Hz.
The limit of detectability of current through-casing instruments is 
about 0.003 A/m², and we obtain results of amplitude above 10⁻¹¹ V/m². 

We have studied several relevant physical effects numerically that 
occur in resistivity logging through casing by utilizing a self-adaptive, 
goal-oriented, hp-FEM. Our software delivers results with 
guaranteed numerical errors below 1%. Thus, we can accurately 
measure both large and small (1%–5%) variations corresponding to 
physical effects associated with through-casing resistivity measure-
ments.

The simulations performed indicate various physical conclusions 
about through-casing resistivity measurements. First, as we increase 
the frequency (from 3 to 90 Hz), the received signal decreases be-
cause of the high casing conductivity and the faster decay of the EM 
fields at higher frequencies, whereas phase variations increase, as 
physically expected. Thus, high frequencies lead to measurements 
below the limit of detectability of actual logging instruments. On the 
other hand, if we decrease the frequency from 3 to 0.3 Hz, sensitivi-
ty with respect to spatial variations of electrical conductivity of the 
formation remains equal in terms of the amplitude of the measure-
ments (indicating we are in a quasi-electrostatics regime) and dimin-
ishes in terms of the phase of the measurements, thereby limiting 
the actual sensitivity of the measurements to the formation resistivity. 
Hence, we recommend the use of frequencies within the 3–30 Hz 
range.

Water invasion effects can be sensed clearly through casing. Vari-
atations caused by water invasion on the model problems we consid-
ered are large (up to one order of magnitude in terms of the ampli-
tude). Moreover, the phase of the measurements seems adequate to 
monitor water invasion.

From the performed simulation, we also infer that it is possible to 
identify thin layers of shales and sands in the formation, provided 
that sand layers have not been invaded with water of resistivity simi-
lar to that of the shale.

Finally, we conclude that anisotropy effects produce nonmeasur-
able variations in our model problem. Thus, detectability of aniso-
tropic formations in cased wells is not possible with coaxial toroidal 
coils.

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APPENDIX A

VALIDATION OF RESULTS

We validate our software by comparing the self-adaptive, goal-
oriented, hp-FEM with a radial code.

A radial code is a software that utilizes a semianalytical method 
based on the Fourier transform. Solutions provided by the radial 
code are identical to the analytical solutions without integration er-
rors, which may be large in problems with highly conductive materi-
als (for example, problems with casing).

On the other hand, numerical errors arising from FEM are of a dif-
ferent nature. We distinguish between modeling errors and approxi-
mation errors. Modeling errors arise because we are solving similar, 
but different, problems to the original ones. These include the use of 
fine computational domains for solving problems on unbounded 
domains. In our case, we consider a computational domain of 
4000 m in the horizontal direction and 8000 m in the vertical direc-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{Simulated through-casing measurements for problem III. (a) Amplitude and (b) phase of the first vertical difference of $E_z$ (normalized) against the vertical position (in meters) of the receiver antennas. Different curves correspond to the presence (or not) of water invasion and possible anisotropy within the shale layers. No invasion, no anisotropy; 15-cm water invasion, no anisotropy; no water invasion and possible anisotropy within the shale layers. No invasion, no anisotropy; 15-cm water invasion, shale anisotropy; 15-cm water invasion, shale anisotropy. Operating frequency: 30 Hz.}
\end{figure}
tion. Another source of modeling errors may be from antenna modeling. To describe the effect of a vertical magnetic dipole (VMD), we consider a solenoidal antenna with finite cross section dimensions of 0.0002 × 0.0002 m and a radius equal to a = 0.003 m with an impressed current I = 1/(πa²) amperes.

Approximation errors arise because we are not solving the modeled problems exactly. We are only approximating them using grids with a finite size h and a finite order of approximation p.

**Model problems considered for the comparison**

Using cylindrical coordinates (ρ, φ, z), we consider the following spatial domains:

\[ \gamma_1 = \{u = (u_\rho, u_\phi, u_z): 0.0^\circ = 0.0000 \ m \leq u_\rho \leq 5.5^\circ = 0.1397 \ m\} \]

\[ \gamma_2 = \{u = (u_\rho, u_\phi, u_z): 5.5^\circ = 0.1397 \ m \leq u_\rho \leq 6.0^\circ = 0.1524 \ m\} \]

\[ \gamma_3 = \{u = (u_\rho, u_\phi, u_z): 6.0^\circ = 0.1524 \ m \leq u_\rho \leq 8.0^\circ = 0.2032 \ m\} \]

\[ \gamma_4 = \{u = (u_\rho, u_\phi, u_z): 8.0^\circ = 0.2032 \ m \leq u_\rho \leq 12^\circ = 0.3048 \ m\} \]

\[ \gamma_5 = \{u = (u_\rho, u_\phi, u_z): 12^\circ = 0.3048 \ m \leq u_\rho\} \]

The source consists of a VMD antenna located at the origin (0,0,0), as shown in Figure A-1. We simulate the electric field response at point (0.13, 0, 1 m).

In Table A-1, we assume two different problems that consider the above-described geometry and different resistivities for each subdomain.

**Comparison results**

Results are compared to each other by computing the following relative errors in percentage:

\[ \text{Error}_1 = \frac{|u^1 - u^2|}{|u^1|} \times 100, \quad (A-1) \]

\[ \text{Error}_2 = \frac{\text{phase}(u^1) - \text{phase}(u^2)}{\text{phase}(u^1)} \times 100, \quad (A-2) \]

\[ \text{Error}_3 = \frac{\text{real}(u^1) - \text{real}(u^2)}{\text{real}(u^1)} \times 100, \quad (A-3) \]

\[ \text{and} \]

\[ \text{Error}_4 = \frac{\text{imag}(u^1) - \text{imag}(u^2)}{\text{imag}(u^1)} \times 100, \quad (A-4) \]

where \( u^1 \) and \( u^2 \) are the solutions provided by the radial code and the FE software, respectively.

Numerical results displaying quantities A-1, A-2, A-3, and A-4 are shown in Figure A-2. Figure A-2a corresponds to problem A-I and
Table A-2. Imaginary part of $E_x$ for problem A-I (in V/m). We compare FE results against those obtained with the radial code with 6400 integration intervals and 3200 integration intervals, respectively.

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Radial code 3200 intervals</th>
<th>Radial code 6400 intervals</th>
<th>FE code</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>$-7.9653E-10$</td>
<td>$-7.9653E-10$</td>
<td>$-7.9683E-10$</td>
</tr>
<tr>
<td>0.1</td>
<td>$-7.9639E-09$</td>
<td>$-7.9639E-09$</td>
<td>$-7.9686E-09$</td>
</tr>
<tr>
<td>1</td>
<td>$-7.8212E-08$</td>
<td>$-7.8212E-08$</td>
<td>$-7.8242E-08$</td>
</tr>
<tr>
<td>10</td>
<td>$-9.8517E-08$</td>
<td>$-9.8516E-08$</td>
<td>$-9.8554E-08$</td>
</tr>
<tr>
<td>$10^2$</td>
<td>$+1.2365E-07$</td>
<td>$+1.2305E-07$</td>
<td>$+1.2309E-07$</td>
</tr>
<tr>
<td>$10^3$</td>
<td>$-4.2700E-09$</td>
<td>$-4.2632E-09$</td>
<td>$-4.2647E-09$</td>
</tr>
<tr>
<td>$10^4$</td>
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<td>$-6.7217E-13$</td>
<td>$-6.7335E-13$</td>
</tr>
<tr>
<td>$10^5$</td>
<td>$-5.7236E-12$</td>
<td>$-5.7159E-12$</td>
<td>$-5.7063E-12$</td>
</tr>
<tr>
<td>$10^6$</td>
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<td>$-5.3760E-11$</td>
</tr>
<tr>
<td>$10^7$</td>
<td>$-4.0959E-11$</td>
<td>$-3.5861E-11$</td>
<td>$-3.8115E-11$</td>
</tr>
</tbody>
</table>

Figure A-3. Problem A-I. Azimuthal component of the electric field $E_x$ (in volts/meter) as a function of frequency. We display the real (solid line and triangles) and imaginary (dashed line and circles) parts of the solutions provided by the radial (triangles and circles) and FE (solid and dashed lines) codes.

Figure A-2b to problem A-II. In all cases, we observe excellent agreement between solutions obtained with the radial code and solutions obtained with the hp-FEM software. More precisely, the relative error remains below 0.1% in most cases. We also observe a slight increase of the error as we increase the frequency. This error increase is because of integration errors on the radial code. A limit of 6400 integration intervals with a Gaussian integration rule of degree 8 has been imposed as the maximum accuracy for integration on the radial code. Numerical results indicate that this limit is not enough at high frequencies to obtain an error below 0.1%. Table A-2, corresponding to problem A-I, shows that the radial code has not fully converged on the imaginary part at 10 MHz.

In Figure A-3, we display the solution of problem A-I as a function of frequency. Figure A-4 shows the solution of problem A-I as a function of the resistivity of layer 2 at 1 Hz and at 1 Mhz.

In summary, we show in this appendix that numerical errors associated with our FE software are negligible (below 0.1%).

REFERENCES


