Simulation of wireline sonic logging measurements acquired with Borehole–Eccentered tools using a high-order adaptive finite-element method

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1. Introduction

While most acoustic logging instruments incorporate stabilizers to center the tool in a borehole, it is often impossible to completely eliminate the effect of borehole eccentricity in sonic measurements. Simulations of measurements acquired with borehole-eccentered sonic logging instruments are critical to understand and interpret acoustic measurements acquired in actual logging conditions. Due to the complexity involved in these type of numerical simulations, previous related work is rather limited. In particular, we highlight the pioneering works of [1–6] in the last part of the twentieth century, and some numerical results for logging-while-drilling (LWD) tools [7,8], for dipole, and quadrupole source excitation [9].

This paper introduces a robust, efficient, and accurate simulation method based on a Fourier series expansion over a particular system of coordinates combined with a two-dimensional (2D) hp-adaptive finite-element method (FEM) [10,11], where $h$ denotes the element size and $p$ the polynomial order of approximation, both varying locally throughout the grid. A similar method was successfully applied to simulate induction and laterolog resistivity measurements in deviated wells with possibly borehole-eccentered logging instruments [12,13]. Our main contribution is the extension of the method to
the case of acoustic measurements. We note that the numerical simulation of borehole sonic measurements is more complex than the simulation of borehole resistivity measurements owing to the inherent multiphysics nature of the former (acoustics coupled with elasticity), and to the fact that propagation media are no longer diffusive, thereby requiring a method for truncation of the computational domain. We employ a perfectly matched layer (PML) for this purpose.

The high accuracy of the new simulation method is confirmed by comparing numerical results against the exact solution for a simple elasto-acoustic borehole model problem. We note that the semi-analytical solution developed for this purpose is another important contribution stemming from this paper.

Our numerical simulations also illustrate the main physical principles underlying acoustic measurements acquired with borehole-eccentered tools. Specifically, we show that higher-order wave propagation modes appear as a direct consequence of tool-eccentricity.

The paper is organized as follows: First, we introduce the three-dimensional finite-element formulation of the problem. Second, we describe our hp-adaptive Fourier finite-element method. Third, we outline the derivation of the semi-analytical solution that is used for verification purposes. Implementation details are described before illustrating the high accuracy of our method with verification and numerical results. Finally, we summarize the main results stemming from this work and describe future extensions. The paper also includes an appendix which explains why the same simulation method fails when applied to deviated wells due to lack of convergence associated with the PML.

2. Method

In this section, we outline the main features of our method. First, we introduce the three-dimensional variational formulation. Then, we describe a new system of coordinates for borehole-eccentered logging tools, which also includes a perfectly matched layer (PML) method for truncation of the computational domain. The method is completed by performing a Fourier transform over the quasi-azimuthal component in a new system of coordinates, and by employing a 2D self-adaptive hp-FEM over the remaining spatial variables (see [10,11]).

2.1. Variational formulation in 3D

The elasto-acoustic problem governing the physics of sonic logging measurements can be expressed in its variational (weak) form in terms of the pressure \(p\) and the displacement \(u\) as

\[
\begin{align*}
\text{Find } (p, u) &\in H^1_0(\Omega_0) \times (H^1_0(\Omega_0))^3 \text{ such that:} \\
\langle
\nabla \cdot \vec{v}, p \rangle_{L^2(\Omega_0)} - \int_{\Gamma_I} \int_{\Gamma_I} \nu \beta \langle q, \nu \cdot \mathbf{u} \rangle_{L^2(\Gamma_I)} &= \langle q, f_0 \rangle_{L^2(\Omega_0)} \\
\langle \mathbf{e}(v) \cdot 
abla \mathbf{u}, f \rangle_{L^2(\Omega_0)} - \int_{\Gamma_I} \int_{\Gamma_I} \nu \beta \langle \mathbf{v}, \rho_1 \mathbf{u} \rangle_{L^2(\Gamma_I)} + \langle \mathbf{n} \cdot \mathbf{v}, f \rangle_{L^2(\Gamma_I)} &= \langle \mathbf{v}, f_i \rangle_{L^2(\Gamma_I)} \\
&\forall q \in H^1_0(\Omega_0), u \in (H^1_0(\Omega_0))^3,
\end{align*}
\]

where \(\Omega_0\) is the acoustic part of the domain, \(\Omega_0\) is the elastic part of the domain, \(\Gamma_I\) is the interface between the acoustic and elastic parts, \(\nu = \omega / \nu_f\) is the wavenumber in a fluid, \(\mathbf{n}_f\) and \(\mathbf{n}_e\) are the normal (outward) vectors with respect to the fluid and solid, respectively, \(\epsilon\) denotes the strain, \(C\) is a fourth-order stiffness tensor, \(\rho_1\) and \(\rho_2\) are the solid and fluid densities, respectively, and \(f_0\) and \(f_i\) are the acoustic and elastic forces, respectively. In the above equation we denote the \(L^2(\Omega)\) inner-product of two (possibly complex and vector-valued) functions \(r\) and \(s\) as

\[
\langle r, s \rangle_{L^2(\Omega)} = \int_{\Omega} r^* s dx_1 dx_2 dx_3,
\]

where \(r^*\) denotes the complex conjugate of \(r\).

2.2. Change of coordinates

In this subsection, we define the mapping \(\psi\) that determines the change of coordinates \(\zeta = \psi(x)\). As illustrated in Fig. 1, this mapping is the composition of three different mappings:

\[
\psi = \psi_{\text{PHY}} \circ \psi_{\text{PML}} \circ \psi_{\text{REF}}.
\]

where \(\psi_{\text{PHY}}\) is the mapping from the canonical domain (in a vertical tool-centered well) to the physical domain, \(\psi_{\text{PML}}\) is the mapping from the canonical domain without a PML into the canonical domain with a PML, and \(\psi_{\text{REF}}\) is the mapping from the reference element of integration to the canonical domain without a PML.

Using the chain rule on this composition of mappings, the Jacobian matrix of the mapping is

\[
\mathcal{J} = \mathcal{J}_{\text{PHY}} \mathcal{J}_{\text{PML}} \mathcal{J}_{\text{REF}}.
\]

where \(\mathcal{J}\) is the Jacobian matrix associated with \(\psi\), \(\mathcal{J}_{\text{PHY}}\) is the Jacobian matrix associated with \(\psi_{\text{PHY}}\), \(\mathcal{J}_{\text{PML}}\) is the Jacobian matrix associated with \(\psi_{\text{PML}}\), and \(\mathcal{J}_{\text{REF}}\) is the Jacobian matrix associated with \(\psi_{\text{REF}}\).
2.2.1. Change of coordinates from the Canonical domain to the physical domain: $\psi_{PHY}$

Let $\phi$ be the azimuthal angle of eccentricity. We define our change of coordinates as

\[
\begin{align*}
\begin{cases}
x_1 &= f_1(\zeta_1) \cos \phi + \zeta_1 \cos \zeta_2, \\
x_2 &= f_1(\zeta_1) \sin \phi + \zeta_1 \sin \zeta_2, \\
x_3 &= \zeta_3, 
\end{cases}
\end{align*}
\]

where $f_1$ is given by the formula

\[
f_1(\zeta_1) = f_1 = \begin{cases}
\rho_0 & \zeta_1 < \rho_1, \\
\frac{\rho - \rho_0}{\rho_1 - \rho_0} \rho_0 & \rho_1 \leq \zeta_1 \leq \rho_2, \\
0 & \zeta_1 > \rho_2
\end{cases}
\]

and $\rho_0$, $\rho_1$, and $\rho_2$ are defined in Fig. 2.

2.2.2. Change of coordinates from the Canonical domain without a PML to the Canonical domain with a PML: $\psi_{PML}$

Following the interpretation of a PML as a change of variables in the complex plane (first proposed by [14] and further discussed in [15]), we define our PML change of coordinates as

Fig. 1. The mapping from the reference element to the physical element is a composition of three mappings.

Fig. 2. Left panel: top view of the geometry describing the location of a sonic tool eccentered in the borehole. The radius of the logging instrument is equal to $\rho_1$, and the radius of the borehole is equal to $\rho_2$, with the distance between the center of the logging instrument and the center of the borehole equal to $\rho_0$. Right panel: iso-lines corresponding to the change of coordinates for borehole-eccentered measurements described in Eq. (5), with $\rho_0 = 0.3$, $\rho_1 = 0.5$, and $\rho_2 = 1$.
2.3. Fourier series expansion

In the acoustic part of the domain, using the new system of coordinates \( r^{\text{PML}} \), and making use of the chain rule, we obtain

\[
\mathbf{v}_p = \mathbf{J}^{-1} \mathbf{v}^{\ddagger} \tilde{p},
\]

where we denote \( \tilde{p} := p \circ \psi = \tilde{p}(\zeta) \), \( \mathbf{v}^{\ddagger} := \mathbf{v}^{\ddagger}(\zeta) \), and \( \mathbf{J} := \begin{pmatrix} A_{1} & A_{2} & A_{3} \end{pmatrix} \). Thus,

\[
\langle \mathbf{v}_q, \mathbf{v}_p \rangle_{L^2(\Omega_i)} = \left\langle \mathbf{v}^{\ddagger} q, \mathbf{J}^{-1} \mathbf{J}^{-1} \mathbf{v}^{\ddagger} p \right\rangle_{L^2(\Omega_i)} = \left\langle \mathbf{v}^{\ddagger} q, A_i \mathbf{v}^{\ddagger} p \right\rangle_{L^2(\Omega_i)},
\]

where \( \mathbf{J} := \text{det}(\mathbf{J}), A_i := |\mathbf{J}| \mathbf{J}^{-1} \mathbf{J}^{-1} \), and the \( L^2(\Omega) \)-inner product in the new system of coordinates \( \zeta \) is defined as:

\[
\langle q, p \rangle_{L^2(\Omega)} = \int_{\Omega} q^{\ddagger} p^{\ddagger} d\zeta_1 d\zeta_2 d\zeta_3.
\]

Notice that \( \langle q, p \rangle_{L^2(\Omega)} = \langle q, |\mathbf{J}| p \rangle_{L^2(\Omega)} \), where \( \Omega := \Omega \circ \psi \).

Defining \( A_2 := |\mathbf{J}|, \) and \( A_3 := |\mathbf{J}| \) (determinant of the Jacobian matrix defined over the surface \( \Gamma_i \)), then variational formulation (1) in the new system of coordinates \( \zeta \) becomes

\[
\langle \mathbf{v}^{\ddagger} q, A_3 \mathbf{v}^{\ddagger} p \rangle_{L^2(\Omega_i)} - k^2 \langle \tilde{q}, A_2 \tilde{p} \rangle_{L^2(\Omega_i)} - \rho \omega^2 \langle \tilde{q}, A_3 \mathbf{n}_f \cdot \mathbf{u}^{\ddagger} \rangle_{L^2(\Gamma_i)} = \langle q, A_3 \tilde{f} \rangle_{L^2(\Omega_i)},
\]

where \( \tilde{f} := f \circ \psi \).

2.3. Fourier series expansion

We observe that all terms in our variational formulations (see, for example, (12)) can be expressed as

\[
\langle \tilde{g}, M \rangle_{L^2(\Omega)}
\]

where \( \tilde{g} \) are arbitrary scalar (or vector) valued functions of \( \zeta \), and \( M \) is either a scalar or a tensor.

Taking the Fourier series expansion of \( M \), and using the definition of inverse Fourier series expansion, we obtain

\[
M = \sum_{l=-\infty}^{\infty} \mathcal{F}_l(M) e^{i 2 \pi l}.
\]
where \( \mathcal{F}_l(M) \) is the \( l \)th Fourier mode of \( M \). Similarly, for \( \tilde{f} \) we obtain
\[
\tilde{f} = \sum_{l} \mathcal{F}_l(\tilde{f}) e^{i2\pi l x}.
\] (15)

Using the above equalities, and the \( L^2 \)-orthogonality of the Fourier basis \( \{ e^{i2\pi l x} \}_{l=-\infty}^{\infty} \), term (13) for each mono-modal test function \( \mathcal{F}_l(\tilde{g}) e^{i2\pi l x} \) is expressed as:
\[
\left\langle \mathcal{F}_l(\tilde{g}) e^{i2\pi l x}, Mf \right\rangle_{L^2(\Omega_{2D})} = 2\pi \sum_{n=-\infty}^{\infty} \left\langle \mathcal{F}_l(\tilde{g}), \mathcal{F}_{l-n}(M) \mathcal{F}_n(f) \right\rangle_{L^2(\Omega_{2D})},
\] (16)
where \( \Omega_{2D} \) is a 2D cross-section of domain \( \Omega \), which is assumed to be axisymmetric.

2.3.1. Acoustics

By substituting equality (16) into Eq. (12) and dividing by \( 2\pi \), we obtain the following variational formulation in terms of Fourier coefficients for each mono-modal test function \( \mathcal{F}_l(\tilde{q}) e^{i2\pi l x} \):
\[
\text{Find } \tilde{p} = \sum_n \mathcal{F}_n(\tilde{p}) e^{i2\pi l x} \text{ such that:}
\]
\[
\sum_{n=-\infty}^{\infty} \left\langle \mathcal{F}_l(\tilde{V} \tilde{q}), \mathcal{F}_{l-n}(A_1) \mathcal{F}_n(\tilde{p}) \right\rangle_{L^2(\Omega_{2D,A})} - k^2 \sum_{n=-\infty}^{\infty} \left\langle \mathcal{F}_l(\tilde{q}), \mathcal{F}_{l-n}(A_2) \mathcal{F}_n(\tilde{p}) \right\rangle_{L^2(\Omega_{2D,A})} - \rho \alpha^2 \sum_{n=-\infty}^{\infty} \left\langle \mathcal{F}_l(\tilde{q}), \mathcal{F}_{l-n}(A_3) \mathcal{F}_n(\mathbf{n} \cdot \tilde{u}) \right\rangle_{L^2(\Omega_{2D,i})} = \sum_{n=-\infty}^{\infty} \left\langle \mathcal{F}_l(\tilde{q}), \mathcal{F}_{l-n}(A_2) \mathcal{F}_n(\tilde{f}_A) \right\rangle_{L^2(\Omega_{2D,A})} \forall l.
\] (17)

Notice in the above formulation that
\[
\mathcal{F}_l(\tilde{V} \tilde{q}) = \left( \frac{\partial \mathcal{F}_l(\tilde{q})}{\partial \zeta_1}, j \mathcal{F}_l(\tilde{q}), \frac{\partial \mathcal{F}_l(\tilde{q})}{\partial \zeta_2} \right).
\] (18)

2.3.2. Elasticity

In the elastic part of the domain the situation is much simpler because the borehole-eccentered change of coordinates described in Eq. (5) reduces to cylindrical coordinates. Thus, \( \mathcal{F}_l(M) \) is either \( M = 0 \) or \( 0 \) if \( l \neq 0 \).

2.4. 2D hp-adaptive finite element method

After applying Fourier series expansion along the quasi-azimuthal component \( \zeta_2 \), our variational formulation reduces to a sequence of coupled 2D problems which, by definition, constitute a 3D problem. The main advantage of the proposed method is that this coupling is weak in the sense that each Fourier mode interacts with only a few other Fourier modes. Thus, it creates a sparsity pattern in the resulting stiffness matrix that is exploited by the solver of linear equations in order to minimize the computational requirements.

All Fourier modes are solved over the same grid. The grid is generated by an automatic \( hp \)-adaptive FEM, where \( h \) denotes the element size (height), and \( p \) is the polynomial order of approximation. The \( hp \)-adaptive FEM iterates along the following steps. Given an initial grid, which we refer to as a coarse grid, we construct a fine grid by dividing every element into four smaller elements (\( h \)-refinement), and increasing the polynomial order of approximation by one (\( p \)-enrichment). The difference between the fine and coarse grid solutions is a good estimate of the error function, which is used to guide optimal refinements. More precisely, those elements delivering maximum error are selected for refinement, and the corresponding type of enrichment (either in \( h \) or in \( p \)) is the one that maximizes the error decrease per added unknown. This error decrease is efficiently computed by performing local projections of the error function over combinations of possible \( h \) and \( p \) refinements. Once optimal refinements have been determined, the fine grid solution is discarded, and the newly generated grid is selected as our next optimal coarse grid. This iterative procedure is continued until a user-prescribed error tolerance is achieved. The solution delivered to the user is the final fine grid solution, along with an estimate of the error in the final coarse grid. For details on the \( hp \)-adaptive FEM, we refer to [10,16]. The main advantage of this method is its exponential convergence in terms of the energy norm versus the problem size (see [17]). An example of a final \( hp \)-grid for a sonic logging problem is shown in Fig. 3, where we observe different element sizes and orders of approximation throughout the computational domain.

Since the physics underlying sonic logging is based on a wave propagation problem, the energy of the solution is (almost) evenly distributed throughout the entire computational domain. Thus, the use of an energy-norm based adaptive strategy as the one proposed here provides adequate results at the receivers, avoiding the need for goal-oriented adaptive algorithms.

Spatial counterparts are then expressed as formal inverse Fourier transforms:

\[ T_n = \text{formal inverse Fourier transform} \]

where \( T_n \) is the Fourier transform of \( p \), and \( \delta(\cdot) \) is the Dirac delta function.

3. Semi-analytical solution

In this section, we consider a fluid-filled borehole without sonic tool, surrounded by a homogeneous isotropic elastic media. We begin computing the acoustic response of the borehole due to a time-harmonic point source excitation located inside the fluid. Then, by integrating the source variable over a circumference, we are able to reproduce a ring source at any location within the borehole. The observation variable is also integrated over the same circumference but at different heights to reproduce the receiver response.

3.1. Time-harmonic point source response

We use the cylindrical coordinates system \((r, \theta, z)\) in \( \mathbb{R}^3 \). We start with a point source placed at \((r_s, 0, 0)\), where \( 0 \leq r_s < \rho_2 \).

The pressure response in the fluid domain is governed by the Helmholtz equation

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p &= -\frac{1}{r} \delta(r - r_s) \delta(\theta) \delta(z) \quad \text{in} \quad \Omega_f, \\
\frac{\partial p}{\partial r} &= \rho_0 \omega^2 \mathbf{u} \cdot \hat{r} \quad \text{at} \quad r = \rho_2, \\
\hat{r} &\cdot \mathbf{u} = 0 \quad \text{as} \quad |z| \to +\infty.
\end{align*}
\]

\( \Omega_f = \{(r, \theta, z) : r < \rho_2\} \). The symbol \( \delta \) denotes the one-dimensional Dirac delta distribution and \( \hat{r} \) stands for the radial unit vector. Equations for the displacement field \( \mathbf{u} \) in the surrounding elastic medium are given by

\[
\begin{align*}
\rho_0 \omega^2 \mathbf{u} + \text{div} \mathbf{\sigma} &= 0 \quad \text{in} \quad \Omega_E = \{(r, \theta, z) : r > \rho_2\}, \\
\mathbf{\sigma} \cdot \hat{r} &= -p \hat{r} \quad \text{at} \quad r = \rho_2, \\
\text{Radiation Condition} &\quad \text{when} \quad ||(r, \theta, z)|| \to +\infty,
\end{align*}
\]

where \( \mathbf{\sigma} \) denotes the stress tensor.

We apply a Fourier transform in \((\theta, z)\) to the Eqs. (19) and (20), i.e.,

\[
\begin{align*}
\tilde{p}_n &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \cos(n\theta)p e^{izd} d\theta dz, \\
\tilde{u}_n &= \frac{1}{2\pi} \int_{\mathbb{R}^2} T_n(\theta) \mathbf{u} e^{izd} d\theta dz,
\end{align*}
\]

where \( T_n(\theta) := \text{diag}(\cos(n\theta), -\sin(n\theta), \cos(n\theta)) \). The remaining system of ordinary differential equations in the radial variable can be solved analytically for \( \tilde{p}_n \) and \( \tilde{u}_n \) in terms of Bessel functions of \( n \)th order. The method is standard, c.f., [18]. The spatial counterparts are then expressed as formal inverse Fourier transforms:

\[
\begin{align*}
p &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \epsilon_n \cos(n\theta) \tilde{p}_n e^{-izd} d\theta dz, \\
\mathbf{u} &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \epsilon_n T_n(\theta) \tilde{u}_n e^{-izd} d\theta dz,
\end{align*}
\]

where \( \epsilon_n = 1 \) if \( n = 0 \) and \( \epsilon_n = 2 \) otherwise. For a general point source \((r_s, \theta_s, z_s) \in \Omega_n\), we need to replace \( \theta \) by \( \theta - \theta_s \) and \( z \) by \( z - z_s \) in the expressions (22).

Semi-analytical techniques are used to perform the Fourier inversion of \( \tilde{p}_n \) in (22). Specifically, \( \tilde{p}_n \) possesses singularities which renders a direct numerical Fourier transform highly unstable. Analogously to [19], by linearity arguments we isolate and treat analytically the singular contributions originating from:

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• Poles associated with pseudo-Rayleigh \((n = 0)\) and flexural \((n \geq 1)\) modes.
• Slow decay at infinity, associated with the Fourier transform of the fundamental solution of the Helmholtz operator in \(\mathbb{R}^3\).
• A log-type singularity due to the presence of the Hankel function \(H^n_0(\cdot)\) in the analytical expression of \(\rho_n\) for \(n = 0\).

Particularly helpful are the inversion formulas that can be found in [20].

After removing all the singular contributions listed above, the regular remaining part is treated numerically with an efficient FFT technique.

### 3.2. Ring source/receiver

In a second step, we consider a ring of radius \(\rho_1\) centered at the point \((\rho_0, 0, 0)\) in \(\Omega_h\), such that \(\rho_0 + \rho_1 < \rho_2\). The points over the ring are described using the curve in polar coordinates defined parametrically for \(0 \leq t < 2\pi\) by

\[
 r(t) = \sqrt{\rho_0^2 + \rho_1^2 + 2\rho_0\rho_1\cos t} \quad \text{and} \quad \theta(t) = \arctan \left( \frac{\rho_0 + \rho_1 \cos t}{\rho_1 \sin t} \right).
\]

Using the parametrization (23), we compute \(p\) (eq. (22)) with the procedure outlined in Section 3.1 for some proper selection of discrete source and observation points:

\[
 \{(r_i(s_i), \theta_i(s_i), 0)\} \quad \text{and} \quad \{(r_i(t_i), \theta_i(t_i), i), 0\}, \quad \text{with} \quad 0 \leq s, i < 2\pi.
\]

Finally, we implement a numerical quadrature to evaluate the integral

\[
P(z) = \rho_1^3 \int_0^{2\pi} \int_0^{2\pi} p((r(t), \theta(t), z); (r_i(s), \theta_i(s), 0)) ds dt.
\]

### 4. Implementation

We implement our method as a Fourier extension of the 2D \(hp\)-adaptive FEM. In this Section, we first describe our 2D multiphysics \(hp\) self-adaptive FEM. Then, we present its extension to 3D by using a Fourier series expansion in the third (quasi-azimuthal) spatial dimension.

#### 4.1. 2D \(hp\)-adaptive finite element method (FEM)

Our 2D \(hp\)-FEM enables the use of a different number of equations per node. Thus, in the acoustic (fluid) part of the domain, only one scalar equation is needed to solve for pressure. In the elastic (solid) part of the domain, we solve for displacement using one vector equation. On the interface between the acoustic and elastic parts of the domain, we employ two equations: one in terms of pressure and another one in terms of displacement.

Because our algorithm enables the use of \(H^1(\text{curl}), H(\text{div})\), and \(L^2\) degrees of freedom, we could have explored different formulations for our elasto-acoustic problem (such as solving in terms of the velocity and the stresses). However, and for simplicity, in this work we have limited ourselves to a formulation in terms of pressure and displacement, which only requires the use of \(H^1\)-spaces.

An important feature of our 2D algorithm is the \(hp\)-adaptive strategy, which automatically produces a sequence of optimal \(hp\)-grids from a given initial grid.

#### 4.2. Fourier extension

In order to implement a Fourier series expansion in the third (quasi-azimuthal) spatial direction, we have allocated a new dimension to each variable of the 2D code: the size of this new dimension is the number of modes in the Fourier series expansion. This number can be dynamically modified during execution time and, for simplicity, we have restricted ourselves to the case in which the number of Fourier modes is constant throughout the entire 2D grid.

Due to the change of coordinates selected in Eq. (5), coupling terms among different Fourier modes occur only in the acoustic part of the domain, which simplifies the implementation of the elastic equations.

### 5. Numerical results

We first perform a detailed numerical verification of the implemented software. Then, we describe numerical simulations for a wireline sonic tool in a borehole environment for fast and slow formations.

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Fig. 4. Comparison of the numerical solution in terms of the magnitude of the normalized pressure to the analytical solution for a 2D sonic logging problem in an open borehole.

Fig. 5. Convergence history for a homogeneous pure acoustic problem as we increase the number of Fourier modes. Different curves correspond to different distances between the center of sources/receivers and the axis of symmetry. We consider four different cases: (a) Top-left panel: 4 kHz, $V_p = 1524$ m/s, (b) Top-right panel: 10 kHz, $V_p = 1524$ m/s, (c) Bottom-left panel: 20 kHz, $V_p = 1524$ m/s, and (d) Bottom-right panel: 10 kHz, $V_p = 3048$ m/s.
5.1. Verification

5.1.1. 2D problem

Fig. 4 compares the numerical solution versus the analytical one when considering a monopole source in a fast formation with no logging instrument (open borehole). In this comparison, the center of source/receivers coincides with the axis of symmetry of the 2D problem. We observe that both solutions coincide for all frequencies.

Fig. 6. Comparison of the numerical solution to the analytical one for a borehole-eccentered sonic logging problem at 5 kHz. Different panels correspond to various distances between the center of sources/receivers and the axis of symmetry.

Fig. 7. Geometrical description of a model wireline logging instrument in a borehole environment.

5.1. Verification

5.1.1. 2D problem

Fig. 4 compares the numerical solution versus the analytical one when considering a monopole source in a fast formation with no logging instrument (open borehole). In this comparison, the center of source/receivers coincides with the axis of symmetry of the 2D problem. We observe that both solutions coincide for all frequencies.
5.1.2. Pure acoustics case

Next, we invoke a homogeneous pure acoustic problem—no elasticity—, and we place sources and receivers off-center. Since the medium is homogeneous, the exact solution is independent of the distance between the axis of symmetry and the center of sources and receivers. From the numerical point-of-view, to accurately solve the problem we will need a larger number of Fourier modes as we increase the distance between sources/receivers and the axis of symmetry. This behavior can be observed in Fig. 5. Results clearly indicate that the implementation in terms of multiple Fourier modes is reliable.

5.1.3. Comparison of numerical simulation results against semi-analytical solution

In our final verification example, we compare the solution of a problem in an open borehole versus the semi-analytical solution described in Section 3. This comparison is performed for a coupled sonic logging problem (acoustics + elasticity) in an open borehole without a logging instrument at a 5 kHz frequency. Fig. 6 shows that the numerical solution is extremely accurate. We emphasize that this type of accuracy is extremely difficult to achieve with traditional methods. Furthermore, we have observed that other codes providing accurate dispersion curves fail to deliver a good agreement for a particular frequency with respect to the exact solution. The high accuracy of our method endows us with an additional level of confidence on the simulation results. This is crucial when designing new logging instruments.

5.2. Numerical simulations of wireline sonic instruments

We describe numerical results corresponding to borehole-eccentered wireline sonic logging measurements in both soft and hard formations. In both cases, we consider the wireline problem sketched in Fig. 7 with a monopole source and the material properties described in Table 1.

### Table 1
Assumed material properties.

<table>
<thead>
<tr>
<th></th>
<th>( \rho ) (kg/m(^3))</th>
<th>( V_p ) (m/s)</th>
<th>( V_s ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fast Formation</td>
<td>2300</td>
<td>4354</td>
<td>2620</td>
</tr>
<tr>
<td>Slow Formation</td>
<td>2100</td>
<td>2540</td>
<td>1269</td>
</tr>
<tr>
<td>Fluid</td>
<td>1100</td>
<td>1524</td>
<td>–</td>
</tr>
<tr>
<td>Tool</td>
<td>5900</td>
<td>5862</td>
<td>2519</td>
</tr>
</tbody>
</table>

Fig. 8. Dispersion curves for a wireline sonic logging instrument in a fast formation with a centered tool (top-left), 1 cm borehole-eccentered tool (top-right), 5 cm borehole-eccentered tool (bottom-left), and the Stoneley mode for various borehole-eccentered distances.

5.1.2. Pure acoustics case

Next, we invoke a homogeneous pure acoustic problem—no elasticity—, and we place sources and receivers off-center. Since the medium is homogeneous, the exact solution is independent of the distance between the axis of symmetry and the center of sources and receivers. From the numerical point-of-view, to accurately solve the problem we will need a larger number of Fourier modes as we increase the distance between sources/receivers and the axis of symmetry. This behavior can be observed in Fig. 5. Results clearly indicate that the implementation in terms of multiple Fourier modes is reliable.

5.1.3. Comparison of numerical simulation results against semi-analytical solution

In our final verification example, we compare the solution of a problem in an open borehole versus the semi-analytical solution described in Section 3. This comparison is performed for a coupled sonic logging problem (acoustics + elasticity) in an open borehole without a logging instrument at a 5 kHz frequency. Fig. 6 shows that the numerical solution is extremely accurate. We emphasize that this type of accuracy is extremely difficult to achieve with traditional methods. Furthermore, we have observed that other codes providing accurate dispersion curves fail to deliver a good agreement for a particular frequency with respect to the exact solution. The high accuracy of our method endows us with an additional level of confidence on the simulation results. This is crucial when designing new logging instruments.

5.2. Numerical simulations of wireline sonic instruments

We describe numerical results corresponding to borehole-eccentered wireline sonic logging measurements in both soft and hard formations. In both cases, we consider the wireline problem sketched in Fig. 7 with a monopole source and the material properties described in Table 1.

5.2.1. Fast formation

For a wireline tool in a fast formation, we observe in Fig. 8 that the main modes are almost independent of the distance between the center of the tool and borehole. Nevertheless, the effect of tool eccentricity is inferred from the fact that new higher order (flexural) modes appear as we increase the eccentricity distance.

5.2.2. Slow formation

For a wireline tool operating in a soft formation, we observe in Fig. 9 that the Stoneley mode is masked by a new dipole tool mode appearing as a result of tool eccentricity. Nevertheless, both the low-frequency and high-frequency asymptotes of the Stoneley mode remain insensitive to the distance from the center of the borehole to the axis of symmetry. Fig. 10 displays the corresponding waveforms for a centered and 5 cm borehole-eccentered tool. In the latest case we clearly observe the appearance of a new dipole tool mode.

6. Conclusions

We developed and successfully tested a new method for simulation of borehole-eccentered sonic logging measurements. The method is based on a change of coordinates followed by a Fourier series expansion in a quasi-azimuthal direction, combined with a 2D $hp$-adaptive FEM in the remaining spatial dimension. A PML was used for truncation of the computational domain. We also developed a semi-analytical solution for verification purposes. Numerical results confirmed the high accuracy of the method and the physical reliability of simulated waveforms. In particular, numerical simulations of wireline measurements with monopole source excitation indicate that, in the case of fast rock formations, the main wave propagation modes are unaffected by the tool's eccentricity distance. However, simulations confirmed the presence of new flexural modes which arise when the tool is eccentered. For the case of soft rock formations, we also showed that a flexural wave propagation mode arises when the logging instrument is borehole eccentered.

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Appendix A. A PML convergence problem in deviated wells

This appendix describes a convergence problem that arises in the PML when applying our method to the case of deviated wells. For simplicity, we first restrict ourselves to the case of a homogeneous pure acoustic problem. We consider our hp Fourier Finite Element method after replacing Eq. (5) corresponding to borehole-eccentered tools by the following change of coordinates suitable for deviated wells

\[
\begin{align*}
  x_1 &= \zeta_1 \cos \zeta_2, \\
  x_2 &= \zeta_1 \sin \zeta_2, \\
  x_3 &= \zeta_1 + \theta_0 f_2(\zeta_1) \cos \zeta_2,
\end{align*}
\]

(A.1)

where \( \theta_0 = \tan \theta \) is the dip angle, and \( f_2 \) is defined for given values \( \rho_1 \) and \( \rho_2 \) as

\[
  f_2(\zeta_1) = f_2 = \begin{cases} 
    0 & \zeta_1 < \rho_1, \\
    \frac{\zeta_1 - \rho_1}{\rho_2 - \rho_1} \rho_2 & \rho_1 \leq \zeta_1 < \rho_2, \\
    \frac{\zeta_1}{\zeta_1} & \zeta_1 \geq \rho_2.
  \end{cases}
\]

(A.2)

For a homogeneous pure acoustic problem, the exact solution is independent of the dip angle. Thus, as we increase the number of Fourier modes, the numerical solutions should always converge toward the 2D solution. However, in Fig. A.11 we observe that at a dip angle equal to 40 degrees, we start losing convergence. A similar situation occurs for larger dip angles.

This problem is due to a singularity arising in the PML, as shown in the solution of the central Fourier mode displayed in Fig. A.12. We attribute this problem to the fact that the metric corresponding to the change of coordinates for deviated wells contains nonzero off-diagonal terms, which produce a shift in the direction of propagation of the waves. For the case of a pure acoustic problem, we could possibly re-define the PML to take into account the shift in the direction of propagation of the wave. However, for a more general elasto-acoustic problem in a deviated well, controlling convergence and stability of the PML remains an unsolved challenge, because in addition to the problem described above, we expect group and phase velocities to possibly have different signs, which renders the PML unstable (as described in [21,22]).
Fig. A.11. Convergence history for a homogeneous pure acoustic problem in a deviated well. Different curves correspond to different dip angles.

Fig. A.12. Absolute value of the real part of the pressure for the central Fourier mode on a pure acoustic problem at 10 kHz. Observe the singularity arising in the top-right corner of the figure within the PML region.

References
